

CURVATURE OF HIGHER DIRECT IMAGES

PHILIPP NAUMANN

ABSTRACT. Given a holomorphic family $f : \mathcal{X} \rightarrow S$ of compact complex manifolds and a relative ample line bundle $L \rightarrow \mathcal{X}$, the higher direct images $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ carry a natural hermitian metric. We give an explicit formula for the curvature tensor of these direct images. This generalizes the result of Schumacher [Sch12], where he computed the curvature of $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(K_{\mathcal{X}/S}^{\otimes m})$ for a family of canonically polarized manifolds. For $p = n$, it coincides with a formula of Berndtsson obtained in [Be11]. Thus, when L is globally ample, we reprove his result [Be09] on the Nakano positivity of $f_*(K_{\mathcal{X}/F} \otimes L)$.

1. INTRODUCTION

For an ample line bundle L on a compact complex manifold X of dimension n , the cohomology groups $H^{n-p}(X, \Omega_X^p(L))$ are critical with respect to Kodaira-Nakano vanishing. More generally, we consider the higher direct image sheaves $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ for a proper holomorphic submersion $f : \mathcal{X} \rightarrow S$ of complex manifolds and a line bundle $L \rightarrow \mathcal{X}$ which is positive along the fibers $X_s = f^{-1}(s)$. The understanding of this situation has applications to moduli problems. In his article [Sch12] Schumacher studies the case $L = K_{\mathcal{X}/S}$ where the fiberwise Kähler-Einstein metrics are used to construct a hermitian metric on the relative canonical bundle which turned out to be semi-positive on the total space. A compact curvature formula is given in this case. At first glance, the method of computation seems to be restricted to the Kähler-Einstein situation. In the general case, there is the result [Be09] of Berndtsson about the Nakano (semi-) positivity of the direct image $f_*(K_{\mathcal{X}/S} \otimes L)$ in the case where L is (semi-) positive. His proof relies on a careful choice of representatives of sections and the usage of his “magic formula”¹. Relying on this method of computation, Mourougane and Takayama studied in [MT08] the higher direct images $R^q f_*\Omega_{\mathcal{X}/S}^n(E)$ for a Nakano (semi-) positive vector bundle E over \mathcal{X} . The proof given there relies on an embedding of the higher direct image into a zero’th direct image in order to apply the method of computation given in [Be09]. The authors raised the question whether one can compute the curvature of the higher direct images $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ by using harmonic representatives along the fibers. We address this problem in the present work. The main motivation for this is the observation that Berndtsson’s formula given in [Be11, Th.1.2] coincides with Schumacher’s formula [Sch12, Th.6] in the case $L = K_{\mathcal{X}/S}$. This fact suggests that Schumacher’s method of computation can be carried over to the more general setting. By putting this into practice, the magic will now lie behind the technique of taking Lie derivatives of line bundle valued forms.

2. DIFFERENTIAL GEOMETRIC SETUP AND STATEMENT OF RESULTS

Let $f : \mathcal{X} \rightarrow S$ be a proper holomorphic submersion and (L, h) a line bundle on \mathcal{X} . The curvature form of the hermitian line bundle is given by

$$\omega_{\mathcal{X}} := 2\pi \cdot c_1(L, h) = -\sqrt{-1}\partial\bar{\partial}\log h.$$

We consider the case where the hermitian bundle (L, h) is relative ample, which means that

$$\omega_{X_s} := \omega|_{X_s}$$

2000 *Mathematics Subject Classification.* 32L10, 32G05, 14Dxx.

Key words and phrases. Curvature of higher direct image sheaves, Deformations of complex structures, Families, Fibrations.

¹This terminology was used in [LY14]

are Kähler forms on the n -dimensional fibers X_s . Then one has the notion of a horizontal lift v_s of a tangent vector ∂_s on the base S and we get a representative of the Kodaira-Spencer class

$$A_s := \bar{\partial}(v_s)|_{X_s}.$$

Note that we have $L_{v_s}(\omega_{X_s})^n = 0$ (see [TW04, Be11]). Furthermore, one sets

$$\varphi := \langle v_s, v_s \rangle_{\omega_{\mathcal{X}}},$$

which is called the geodesic curvature. The coherent sheaf $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ is locally free on S outside a proper subvariety. In the case $n = p$ and L ample, the sheaf $f_*(K_{\mathcal{X}/S} \otimes L)$ is locally free by the Ohsawa-Takegoshi extension theorem (see [Be09]). We assume the local freeness of

$$R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$$

in the general case, hence the statement of the Grothendieck-Grauert comparison theorem holds. Now Lemma 2 of [Sch12] applies, which says that we can represent local sections of $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ by $\bar{\partial}$ -closed $(p, n-p)$ -forms on the total space, whose restrictions to the fibers are harmonic. Let $\{\psi^1, \dots, \psi^r\}$ be a local frame of the direct image consisting of such sections around a fixed point $s \in S$. We denote by $\{(\partial/\partial s_i)|i = 1, \dots, \dim S\}$ a basis of the complex tangent space $T_s S$ of S over \mathbb{C} , where s_i are local holomorphic coordinates on S . Let $A_{i\bar{j}}^\alpha(z, s)\partial_\alpha dz^{\bar{\beta}} = \bar{\partial}(v_i)_{X_s}$ be the $\bar{\partial}$ -closed representative of the Kodaira-Spencer class of ∂_i described above. Then the cup product together with contraction defines

$$\begin{aligned} (1) \quad & A_{i\bar{j}}^\alpha \partial_\alpha dz^{\bar{\beta}} \cup : \mathcal{A}^{0, n-p}(X_s, \Omega_{X_s}^p(L|_{X_s})) \rightarrow \mathcal{A}^{0, n-p+1}(X_s, \Omega_{X_s}^{p-1}(L|_{X_s})) \\ (2) \quad & A_{j\bar{\alpha}}^{\bar{\beta}} \partial_{\bar{\beta}} dz^\alpha \cup : \mathcal{A}^{0, n-p}(X_s, \Omega_{X_s}^p(L|_{X_s})) \rightarrow \mathcal{A}^{0, n-p-1}(X_s, \Omega_{X_s}^{p+1}(L|_{X_s})) \end{aligned}$$

where $p > 0$ in (1) and $p < n$ in (2). Note that this is a formal analogy to the derivative of the period map in the classical case (see [Gr70]). We will apply the above cup products to harmonic $(p, n-p)$ -forms. In general, the results are not harmonic.

When applying the Laplace operator to (p, q) -forms with values in L on the fibers X_s , we have

$$(3) \quad \square_\partial - \square_{\bar{\partial}} = (n - p - q) \cdot \text{id}$$

due to the definition $\omega_{X_s} = \omega_{\mathcal{X}}|_{X_s}$ and the Bochner-Kodaira-Nakano identity (see also the proof of Corollary 3). Thus, we write $\square = \square_\partial = \square_{\bar{\partial}}$ in the case $q = n - p$. By considering an eigenfunction decomposition and using the identity (3), we obtain that all eigenvalues of \square are 0 or greater than 1, hence the operator $(\square - 1)^{-1}$ exists. We use the notation $\psi^{\bar{l}} := \overline{\psi^l}$ for sections ψ^l and write $g dV = \omega_{X_s}/n!$. The main result is

Theorem 1. *Let $f : \mathcal{X} \rightarrow S$ be a proper holomorphic submersion and $(L, h) \rightarrow \mathcal{X}$ a relative ample line bundle. With the objects described above, the curvature of $R^{n-p}f_*\Omega_{\mathcal{X}/S}^p(L)$ is given by*

$$\begin{aligned} R_{i\bar{j}}^{\bar{l}k}(s) = & \int_{X_s} \varphi_{i\bar{j}} \cdot (\psi^k \cdot \psi^{\bar{l}}) g dV \\ & + \int_{X_s} (\square + 1)^{-1} (A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \psi^{\bar{l}}) g dV \\ & + \int_{X_s} (\square - 1)^{-1} (A_i \cup \psi^{\bar{l}}) \cdot (A_{\bar{j}} \cup \psi^k) g dV \end{aligned}$$

If $L \rightarrow \mathcal{X}$ is ample, the only contribution, which may be negative, originates from the harmonic parts in the third term

$$- \int_{X_s} H(A_i \cup \psi^{\bar{l}}) \overline{H(A_j \cup \psi^{\bar{k}})} g dV.$$

Corollary 1 (compare [Be09], Th.1.2 and [Be11], Th.1.2). *If $L \rightarrow \mathcal{X}$ is a (semi-) positive line bundle, which is positive along the fibers, then $f_*(K_{\mathcal{X}/S} \otimes L)$ is Nakano (semi)- positive.*

Proof. Because of degree reasons, the third term in Theorem 1 vanishes for $p = n$. The operator $(\square + 1)^{-1}$ is positive. Furthermore, we have

$$\omega_{\mathcal{X}}^{n+1} = \omega_{\mathcal{X}/S}^n \sum \sqrt{-1} \varphi_{i\bar{j}} ds^i \wedge ds^{\bar{j}} = \sum \sqrt{-1} \varphi_{i\bar{j}} \cdot ds^i \wedge ds^{\bar{j}} g dV.$$

Hence, the matrix $(\varphi_{i\bar{j}})$ is positive definite if L is ample. \square

Corollary 2 ([Sch12], Th.6). *If $\mathcal{X} \rightarrow S$ is a family of canonically polarized compact complex manifolds, then the curvature tensor of $R^{n-p} f_* \Omega_{\mathcal{X}/S}^p(K_{\mathcal{X}/S})$ is given by*

$$\begin{aligned} R_{i\bar{j}}^{\bar{l}k}(s) = & \int_{X_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) \cdot (\psi^k \cdot \psi^{\bar{l}}) g dV \\ & + \int_{X_s} (\square + 1)^{-1} (A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \psi^{\bar{l}}) g dV \\ & + \int_{X_s} (\square - 1)^{-1} (A_i \cup \psi^{\bar{l}}) \cdot (A_{\bar{j}} \cup \psi^k) g dV \end{aligned}$$

Proof. The Kähler-Einstein metrics $\omega_{X_s} = \sqrt{-1} g_{\alpha\bar{\beta}}(z, s) dz^\alpha \wedge d\bar{z}^\beta$ on the fibers induce a hermitian metric on the relative canonical bundle $g^{-1} = (\det g_{\alpha\bar{\beta}})^{-1}$ with curvature form $\omega_{\mathcal{X}}$. The Kähler-Einstein condition gives $\omega_{\mathcal{X}}|_{X_s} = \omega_{X_s}$. Furthermore, we have the elliptic equation (see [Sch93])

$$(\square + 1)\varphi_{i\bar{j}} = A_i \cdot A_{\bar{j}}.$$

Note also that the representatives A_i are harmonic in this special case. \square

Corollary 3. *The direct images $R^{n-p} f_* \Omega_{\mathcal{X}/S}^p(L)$ are all Nakano positive, if L is ample and $\mathcal{X} \rightarrow S$ everywhere infinitesimal trivial.*

Proof. If $\mathcal{X} \rightarrow S$ is infinitesimal trivial, we have $A_i = \bar{\partial}(b_i)$ for a differentiable vector field $b_i^\alpha \partial_\alpha$ on the fiber X_s , because A_i represents the Kodaira-Spencer class and hence needs to be $\bar{\partial}$ -exact. The Bochner-Kodaira-Nakano identity says (on the fiber X_s)

$$\square_{\bar{\partial}} - \square_{\partial} = [\sqrt{-1}\Theta(L), \Lambda].$$

But by definition, we have $\omega_{X_s} = \sqrt{-1}\Theta(L)|_{X_s}$. Furthermore, it holds (see [De12, Cor.VI.5.9])

$$[L\omega, \Lambda\omega]u = (p + q - n)u \quad \text{for } u \in \mathcal{A}^{p,q}(X_s, L|_{X_s}).$$

Thus, the $\square_{\bar{\partial}}$ -harmonic $(p, n-p)$ -form $\psi^{\bar{l}}$ is also harmonic with respect to \square_{∂} , in particular ∂ -exact. Therefore,

$$A_i \cup \psi^{\bar{l}} = \bar{\partial}(b_i) \cup \psi^{\bar{l}} = \bar{\partial}(b_i \cup \psi^{\bar{l}}),$$

so the harmonic part of $A_i \cup \psi^{\bar{l}}$ must vanish. \square

3. PROOF OF THE THEOREM

The components of the metric tensor for $R^{n-p} f_* \Omega_{\mathcal{X}/S}^p(L)$ on the base space S is given by ($q = n - p$)

$$\langle \psi^k, \psi^{\bar{l}} \rangle = \int_X \psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^{\bar{l}} g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h g dV,$$

which are integrals of inner products of harmonic representatives of the cohomology classes. These are represented by $\bar{\partial}$ -closed differential forms on the total space. When we compute derivatives with respect to the base of theses fiber integrals, we apply Lie derivatives with respect to the horizontal lifts of the tangent vectors. In order to break up the Lie derivative of the pointwise inner product, we need to introduce Lie derivatives of differential forms with values in a line bundle. This can be done by using the hermitian connection ∇ on $\mathcal{A}^{(p,q)}(X_s, L|_{X_s})$ induced by the Chern connections on (T_{X_s}, ω_{X_s}) and (L, h) . We define the Lie derivative of ψ with respect to the horizontal lift v by using Cartan's formula

$$L_v \psi := (\delta_v \circ \nabla + \nabla \circ \delta_v) \psi$$

and similar for the Lie derivative with respect to \bar{v} . We note that this definition extends the usual Lie derivative for ordinary tensors, which can as well be computed by using covariant differentiation. An important point is the following

Proposition 1.

$$\frac{\partial}{\partial s} \langle \psi^k, \psi^l \rangle = \langle L_v \psi^k, \psi^l \rangle + \langle \psi^k, L_{\bar{v}} \psi^l \rangle.$$

Proof. By Lemma 1 of [Sch12] we have

$$\frac{\partial}{\partial s} \langle \psi^k, \psi^l \rangle = \int_X L_v \left(\psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h g dV \right).$$

The integrand is now a Lie derivative of an ordinary (n, n) -form. We have

$$L_v \left(\psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h g dV \right) = L_v \left(\psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \right) g dV,$$

because $L_v(g dV)$ vanishes. Now the Lie derivative of a function is just the ordinary derivative in the direction of v , so we get (by using Einstein's summation convention and , for ordinary derivatives)

$$\begin{aligned} L_v \left(\psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \right) &= (\partial_s + a_s^\alpha \partial_\alpha) \left(\psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \right) \\ &= (\psi_{A_p \bar{B}_q, s} + a_s^\alpha \psi_{A_p \bar{B}_q, \alpha}) \left(\psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \right) + \psi_{A_p \bar{B}_q}^k (\psi_{C_q \bar{D}_p, s}^l + a_s^\alpha \psi_{C_q \bar{D}_p, \alpha}^l) \left(g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \right) \\ &\quad + \psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l \left(\partial_s (g^{\bar{\delta}_1 \alpha_1}) g^{\bar{\delta}_2 \alpha_2} \dots g^{\bar{\beta}_n \gamma_n} h + g^{\bar{\delta}_1 \alpha_1} \partial_s (g^{\bar{\delta}_2 \alpha_2}) \dots g^{\bar{\beta}_n \gamma_n} h + \dots + g^{\bar{\delta}_1 \alpha_1} g^{\bar{\delta}_2 \alpha_2} \dots \partial_s (g^{\bar{\beta}_n \gamma_n}) h \right) \\ &\quad + \psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l \left(a_s^\alpha \partial_\alpha (g^{\bar{\delta}_1 \alpha_1}) g^{\bar{\delta}_2 \alpha_2} \dots g^{\bar{\beta}_n \gamma_n} h + g^{\bar{\delta}_1 \alpha_1} a_s^\alpha \partial_\alpha (g^{\bar{\delta}_2 \alpha_2}) \dots g^{\bar{\beta}_n \gamma_n} h + \dots + g^{\bar{\delta}_1 \alpha_1} g^{\bar{\delta}_2 \alpha_2} \dots a_s^\alpha \partial_\alpha (g^{\bar{\beta}_n \gamma_n}) h \right) \\ &\quad + \left(\psi_{A_p \bar{B}_q}^k \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} \right) (\partial_s h + a_s^\alpha \partial_\alpha h) \end{aligned}$$

Now we use the identities $\partial_s g^{\bar{\beta} \gamma} = g^{\bar{\beta} \sigma} a_{s; \sigma}^{\bar{\gamma}}$ and $\partial_\alpha g^{\bar{\beta} \gamma} = -g^{\bar{\beta} \sigma} \Gamma_{\alpha \sigma}^{\bar{\gamma}}$ as well as the Christoffel symbols for the Chern connection on (L, h) which are $h^{-1} \partial_s h = \Gamma_s^h$ and $h^{-1} \partial_\alpha h = \Gamma_\alpha^h$. The above somewhat lengthy expression can then be written as (now we use ; for indicating covariant derivatives)

$$\begin{aligned} &(\psi_{A_p \bar{B}_q, s}^k + \Gamma_s^h \psi_{A_p \bar{B}_q}^k + a_s^\alpha (\psi_{A_p \bar{B}_q, \alpha}^k - \sum_{j=1}^p \Gamma_{\alpha \alpha_j}^\sigma \psi_{\alpha_1 \dots \underset{j}{\sigma} \dots \alpha_p \bar{B}_q}^k + \Gamma_\alpha^h \psi_{A_p \bar{B}_q}^k)) \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \\ &\quad + \psi_{A_p \bar{B}_q}^k (\psi_{C_q \bar{D}_p, s}^l + a_s^\alpha (\psi_{C_q \bar{D}_p, \alpha}^l - \sum_{j=p+1}^n \Gamma_{\alpha \gamma_j}^\sigma \psi_{\gamma_{p+1} \dots \underset{j}{\sigma} \dots \alpha_n \bar{D}_p}^l)) g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \\ &+ \left(\sum_{j=1}^p \psi_{\alpha_1 \dots \underset{j}{\alpha} \dots \alpha_p \bar{B}_q}^k a_{s; \alpha_j}^\alpha \right) \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h + \psi_{A_p \bar{B}_q}^k \left(\sum_{j=p+1}^n \psi_{\gamma_{p+1} \dots \underset{j}{\alpha} \dots \gamma_n \bar{D}_p}^l a_{s; \gamma_j}^\alpha \right) g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \\ &= (\psi_{A_p \bar{B}_q, s}^k + a_s^\alpha \psi_{A_p \bar{B}_q, \alpha}^k) \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h + \psi_{A_p \bar{B}_q}^k (\psi_{C_q \bar{D}_p, s}^l + a_s^\alpha \psi_{C_q \bar{D}_p, \alpha}^l) g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \\ &+ \left(\sum_{j=1}^p \psi_{\alpha_1 \dots \underset{j}{\alpha} \dots \alpha_p \bar{B}_q}^k a_{s; \alpha_j}^\alpha \right) \psi_{C_q \bar{D}_p}^l g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h + \psi_{A_p \bar{B}_q}^k \left(\sum_{j=p+1}^n \psi_{\gamma_{p+1} \dots \underset{j}{\alpha} \dots \gamma_n \bar{D}_p}^l a_{s; \gamma_j}^\alpha \right) g^{\bar{D}_p A_p} g^{\bar{B}_q C_q} h \end{aligned}$$

The (p, q) -components of the forms $L_v \psi$ and $L_{\bar{v}} \psi^l$ are given by

$$(L_v \psi^k)_{(p, q)} = (\psi_{A_p \bar{B}_q, s}^k + a_s^\alpha \psi_{A_p \bar{B}_q, \alpha}^k + \sum_{j=1}^p a_{s; \alpha_j}^\alpha \psi_{\alpha_1 \dots \underset{j}{\alpha} \dots \alpha_p \bar{B}_q}^k) dz^{A_p} \wedge dz^{\bar{B}_q}$$

and

$$(L_{\bar{v}} \psi^l)_{(p, q)} = (\psi_{D_p \bar{C}_q, \bar{s}}^l + a_{\bar{s}}^{\bar{\gamma}} \psi_{D_p \bar{C}_p, \bar{\gamma}}^l + \sum_{j=p+1}^n a_{\bar{s}; \bar{\gamma}_j}^{\bar{\gamma}} \psi_{D_p \bar{\gamma}_{p+1} \dots \underset{j}{\bar{\gamma}} \dots \bar{\gamma}_n}^l) dz^{D_p} \wedge dz^{\bar{C}_q},$$

thus the statement of the proposition follows. \square

After removing the obstacles concerning the Lie derivative of line bundle valued forms from one's path, we are now ready to use the method of computation given in [Sch12, Sch13] in the more general setting. The point is that the computation given there carries over verbatim if one sets $m = 1$ and replaces $K_{\mathcal{X}/S}$ by L . One has to check that there is no point where the $\bar{\partial}^*$ -closedness of A_s is used, which is a crucial fact. Moreover, there is no elliptic equation for φ in general. Thus, we must not replace φ by $(\square + 1)^{-1}(|A_s|^2)$. Finally note that by definition

$$\omega_{X_s} = \omega_{\mathcal{X}}|_{X_s}.$$

Then the computation works without the Kähler-Einstein condition.

REFERENCES

- [Be09] B. BERNDTSSON: *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. **169**, 531-560 (2009).
- [Be11] B. BERNDTSSON: *Strict and nonstrict positivity of direct image bundles*, Math. Zeitschrift, Vol. **269**, 1201-1218 (2011).
- [De12] J.-P. DEMAILLY: *Complex Analytic and Differential Geometry*, <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>, Grenoble (2012).
- [Gr70] P.A. GRIFFITHS: *Periods of integrals on algebraic manifolds III: Some differential-geometric properties of the period mapping*, Publ. Math. IHES **38**, 125-180 (1970).
- [LSY04] K. LIU, X. SUN, S.-T. YAU: *Canonical metrics on the moduli space of Riemann surfaces I*, J. Differential Geom. **68**, No. 3, 571-637 (2004).
- [LSY13] K. LIU, X. SUN, X. YANG: *Vanishing theorems for ample vector bundles*, J. Algebraic Geom. **22**, 303-331 (2013).
- [LY14] K. LIU, X. YANG: *Curvature of direct image sheaves of vector bundles and applications*, J. Differential Geom. **98**, No. 1, 117-145 (2014).
- [LSY13] K. LIU, X. SUN, S.-X. YANG: *Vanishing theorems for ample vector bundles*, J. Algebraic Geom. **22**, 303-331 (2013).
- [Ma05] M. MANETTI: *Lectures on deformations of complex manifolds*, arXiv:math/0507286v1 (2005).
- [MK71] J. MORROW, K. KODAIRA: *Complex Manifolds*, Holt, Rinehart and Winston, Inc., New York (1971).
- [MT08] C. MOURougANE, S. TAKAYAMA: *Hodge metrics and the curvature of higher direct images*, Ann. Sci. Ec. Norm. Supér. (4) **41**, 905-924 (2008).
- [Sch93] G. SCHUMACHER: *The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds*, Ancona, V. (ed.) et al., Complex analysis and geometry. New York: Plenum Press. The University Series in Math., 339-354 (1993).
- [Sch12] G. SCHUMACHER: *Positivity of relative canonical bundles and applications*, Invent. Math. **190**, 1-56 (2012).
- [Sch13] G. SCHUMACHER: *Erratum to: Positivity of relative canonical bundles and applications*, Invent. Math. **192**, 253-255 (2013).
- [Siu86] Y.T. SIU: *Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class*, In: H.W. Stoll (ed.) Contributions to Several Complex Variables, Proc. Conf. Complex Analysis, Norte Dame/Indiana, 1984. Aspects Math., vol. E9, pp. 261-298 (1986).
- [TW04] W.-K. TO, L. WENG: *L^2 -Metrics, Projective Flatness and Families of Polarized Abelian Varieties*, Trans. Amer. Math. Soc., **356**, 2685-2707 (2004).

FACHBEREICH MATHEMATIK UND INFORMATIK, PHILIPPS-UNIVERSITÄT MARBURG, LAHNBERGE, HANS-MEERWEIN-STRASSE, D-35032 MARBURG, GERMANY

E-mail address: `naumann@mathematik.uni-marburg.de`